On the geometry of equivariant compactifications of the vector group

(joint works with Z. Huang, B. Fu, A. Dubouloz and T. Kishimoto)

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Recall (Kobayashi-Ochiai): The Fano index of X is the maximum $\iota_X \in \mathbf{N}$ such that $-K_X = \iota_X A$ for some A ample divisor. Moreover, $1 \leq \iota_X \leq n+1$, and $\iota_X = n+1$ (resp. $\iota_X = n$) iff $X \cong \mathbf{P}^n$ (resp. $X \cong \mathbf{Q}^n \subseteq \mathbf{P}^{n+1}$).

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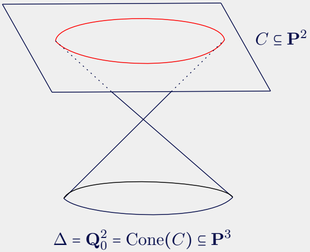
Furushima (1993):

$$(X, \Delta) \cong \begin{cases} (\mathbf{P}^3, \{\text{plane}\}) & (\iota_X = 4) \\ (\mathbf{Q}^3, \mathbf{Q}^2_0) & (\iota_X = 3) \\ (V_5, S_i) \quad i = 1, 2 & (\iota_X = 2) \\ (V_{22}, S_i) \quad i = 1, 2 & (\iota_X = 1) \end{cases}$$



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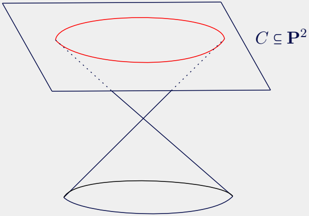

$\Delta = \mathbf{Q}^2_0 = \mathrm{Cone}(C) \subseteq \mathbf{P}^3$

Kuznetsov–Prokhorov–Shramov (2018)

These are the only Fano 3-folds with $\rho(X) = 1$ and infinite $\mathrm{Aut}(X)$.

§2. Additive structures

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A G-**structure** on X is a regular action $G \times X \longrightarrow X$ such that for a general point $x_0 \in X$ we have that:

1. The stabilizer $\mathrm{Stab}(x_0)$ is trivial.
2. The orbit $G \cdot x_0 \cong G$ is dense.

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- $G = \mathbf{G}_a^n = (\mathbf{C}^n, +) \rightsquigarrow X$ **variety with $\mathbf{G}_a^n$-structure**

Arithmetic Geometry:

Northcott (1949): Let $K \supseteq \mathbf{Q}$ be a number field and let $B \in \mathbf{R}^{>0}$. Then, $N(B) = \#\{p \in \mathbf{P}^n(K) \mid H_n(p) \leq B\}$ is finite.

Example ($K = \mathbf{Q}$): Let $p = (x_0, \ldots, x_n) \in \mathbf{Z}^{n+1}$ s.t. $\gcd(x_0, \ldots, x_n) = 1$, then $H_n(p) = \max_{0 \leq i \leq n} |x_i|$ and $N(B) \leq C(n) B^{n+1}$.

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Positivity of the tangent bundle:

If X is a smooth additive variety, then T_X is big (**Liu**, 2023).

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Example ($n = 2$): The algebras $A_i = \mathbf{C}[X,Y]/\mathcal{I}_i \cong_{\mathbf{C}\text{-v.s.}} \mathbf{C}^3$ with

$$\mathcal{I}_1 = \langle X^2, XY, Y^2 \rangle \quad \text{and} \quad \mathcal{I}_2 = \langle XY, Y - X^2 \rangle$$

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Let $(a_1, a_2) \in \mathbf{G}_a^2$, then $\exp([a_1 X + a_2 Y]) \curvearrowright A_i$ induces

$$\rho_1 : [x_0, x_1, x_2] \mapsto [x_0 + a_2 x_2, x_1 + a_1 x_2, x_2] \text{ (naive/toric action)}$$

$$\rho_2 : [x_0, x_1, x_2] \mapsto [x_0 + a_1 x_1 + (a_2 + \frac{1}{2} a_1^2) x_2, x_1 + a_1 x_2, x_2]$$

where ρ_1 (resp. ρ_2) have infinitely many (resp. 3) orbits.

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If X admits a $\mathbf{G}_a^3$-structure, then $X \cong \mathbf{P}^3$ or $\mathbf{Q}^3 \subseteq \mathbf{P}^4$.

Their proof uses the following ingredients:

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- The contradiction[1] comes from a $\mathbf{G}_a^3$-equivariant Sarkisov link $V_5 \dashrightarrow \mathbf{Q}^3$ studied by Furushima–Nakayama (1989). □

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§3. Results

Huang–M., 2020

There are 17 families of smooth additive Fano 3-folds with $\rho(X) \geq 2$. Moreover, all of them can be obtained as:

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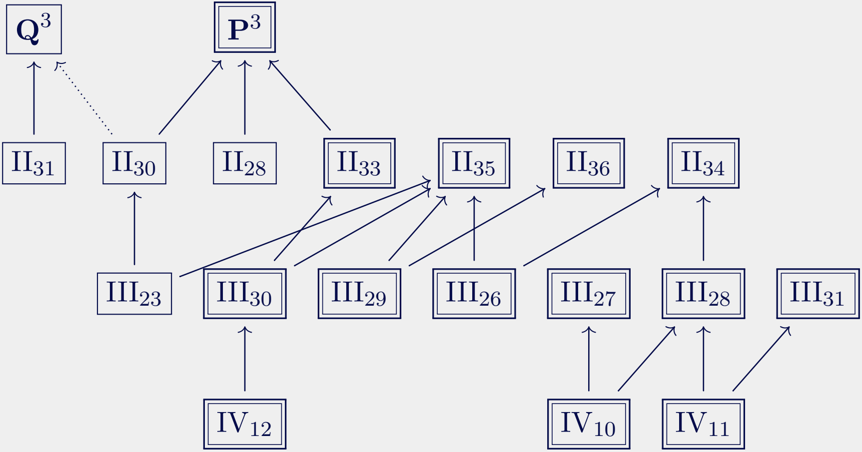
A posteriori, we have that:

- Every such X verifies $\rho(X) \leq 4$
- Every additive Fano 3-fold with $\rho(X) \geq 2$ which is **primitive** (i.e., $X \ncong \mathrm{Bl}_C(Y)$) is toric.



$\mathbf{Q}^3$
$\mathbf{P}^3$
II_{31}
II_{30}
II_{28}
II_{33}
II_{35}
II_{36}
II_{34}
III_{23}
III_{30}
III_{29}
III_{26}
III_{27}
III_{28}
III_{31}
IV_{12}
IV_{10}
IV_{11}

§3. Results (Ingredients for $\dim(X) = 3$)

1. **Iskovskikh y Mori–Mukai:** $\underbrace{17}_{\rho=1} + \underbrace{87 + 1}_{\rho\geq 2}$ Fano threefolds
2. **Blanchard**(**–Brion**)'s Lemma:
   (a) If $\sigma : X \to Y$ blow-up: X additive implies Y additive.
   (b) $\mathrm{Aut}^0(X \times Y) \cong \mathrm{Aut}^0(X) \times \mathrm{Aut}^0(Y)$.

Example:

(a) If $\rho(Y) = 1$ with $Y \ncong \mathbf{P}^3$ nor $\mathbf{Q}^3$ then X is **not** additive.

(b) (**Mori–Mukai**): If $\rho(X) \geq 6$ then $X \cong S_d \times \mathbf{P}^1$ with $1 \leq d \leq 5$. In particular, $\mathrm{Aut}^0(X) \cong \mathrm{PGL}_2(\mathbf{C}) \not\ltimes \mathbf{G}_a^3$

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3. Toric case:
   - **Arzhantsev–Romanskevich** (2017): Combinatorics of additive **toric** varieties
   - **Batyrev** (1982) and **Watanabe-Watanabe** (1982): toric Fano threefolds

Example: $\mathrm{III}_{31} \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus (\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,1))$

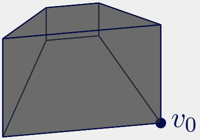

Figure: Fano polytope III_{31}

We got 14 additive toric Fano threefolds. More generally (**Levicán**, 2022):

There are 79/124 (resp. 470/866, resp. 3428/7622) smooth additive toric Fano varieties of dimension 4 (resp. 5, resp. 6).

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4. **Arzhantsev** (2011): $X = G/P$ homogeneous Fano is additive if and only if $R_u(P)$ is commutative (or (G, P) is exceptional).
   Example: $\mathbf{P}(T_{\mathbf{P}^n}) \cong \{x_0y_0 + \ldots + x_ny_n = 0\} \subseteq \mathbf{P}^n \times \mathbf{P}^n$ is **not** additive.
5. **Sharoyko** (2009) and **Arzhantsev–Popovskiy** (2014): Explicit description of the (unique) additive structure of $\mathbf{Q}^n \subseteq \mathbf{P}^n$ and $\mathbf{Q}_0^n = \mathrm{Cone}(\mathbf{Q}^{n-1})$ "à la Hassett-Tschinkel".
6. **Kishimoto** (2005): Classified (X, Δ_1, Δ_2) s.t. $\mathbf{A}^3 \hookrightarrow X$ where X Fano, $\rho(X) = 2$, $X \smallsetminus \mathbf{A}^3 = \Delta_1 \cup \Delta_2$ and additionally

$$(\dagger)\ K_X + \Delta_1 + \Delta_2 \text{ is } \textbf{not } \underline{\textbf{nef}}$$

$\rightsquigarrow$ 16 possible X
$\rightsquigarrow$ We checked that 7 among them are additive. □

§3. Results

$n = \dim(X) \geq 4$:

Fu-M., 2019

We classified all smooth additive Fano n-folds X such that $\iota_X \geq n-2$. In particular, there are 11 families with $\rho(X) = 1$.

Consider $-K_X = \iota_X A$, where $1 \leq \iota_X \leq n+1$:

0. $\iota_X = n+1 \Leftrightarrow X \cong \mathbf{P}^n$ (**Kobayashi–Ochiai**)
1. $\iota_X = n \Leftrightarrow X \cong \mathbf{Q}^n$ (**Kobayashi–Ochiai**)
2. $\iota_X = n-1$ "del Pezzo" (**Fujita**)
3. $\iota_X = n-2$ "Fano–Mukai" (**Mukai, Mella, Wisniewski**)

Two cases:

(a) If $\rho(X) \geq 2$ we use Blanchard's Lemma.

(b) If $\rho(X) = 1$ consider the **VMRT** (**Hwang, Mok, Kebekus**):

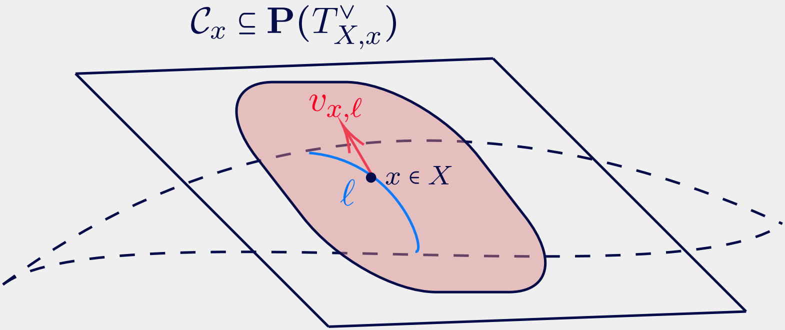

Figure: The VMRT of a Fano manifold X at a general point $x \in X$

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§3. Results

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Fu–M. (2019): Let X be a smooth del Pezzo variety of dimension n admitting a $\mathbf{G}_a^n$-structure. Then,

$$X \cong \begin{cases} \mathrm{Gr}(2,5) \cap L \subseteq \mathbf{P}^9 \text{ linear section} & (\rho(X) = 1,\ d = 5,\ 4 \leq n \leq 6) \\ & \end{cases}$$

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The following condition is conjectured to hold for every smooth additive Fano manifold with $\rho(X) = 1$ (J.-M. Hwang):

($\star$) The VMRT $\mathcal{C}_x \subseteq \mathbf{P}(T^\vee_{X,x})$ of a general point $x \in X$ is smooth

(1°) Fujita's classification: X is isomorphic to

- (a) $X_4 \subseteq \mathbf{P}(2,1,\ldots,1)$ cuartic $\rightsquigarrow \mathrm{Aut}^0(X) = \{1\}$
- (b) $X_3 \subseteq \mathbf{P}^{n+1}$ cubic $\rightsquigarrow \mathrm{Aut}^0(X) = \{1\}$
- (c) Intersection $\mathbf{Q}^n_1 \cap \mathbf{Q}^n_2 \subseteq \mathbf{P}^{n+2} \rightsquigarrow \mathrm{Aut}^0(X) = \{1\}$
- (d) $X_6 \subseteq \mathbf{P}(3,2,1,\ldots,1)$ sextic:
  $\mathrm{Pic}(X) = \mathbf{Z}\mathcal{O}_X(1)$ and $\mathcal{L} = \mathcal{O}_X(1)$ defines a map $\varphi_{\mathcal{L}} : X \dashrightarrow \mathbf{P}^{n-1}$ which is not birational.
  Hwang–Fu (2017): this is impossible.
- (e) Linear section of $\mathrm{Gr}(2,5)$

(2°) **Hwang–Fu** (2017): If X additive with $\rho(X) = 1$, ($\star$) implies $\mathcal{C}_x$ irreducible and linearly non-degenerate ($\Rightarrow \dim(\mathcal{C}_x) \geq 1$).

(3°) We check that if ℓ general minimal rational curve on X additive with $\rho(X)=1$ s.t. $(\star)$, then:

$$\iota_X = -K_X \cdot \ell \; (= \dim(\mathcal{C}_x) + 2 \geq 3)$$

(4°) Condition $(*)$ holds for linear sections of $\mathrm{Gr}(2,5)$, and then $\iota_X = n-1 \geq 3$, i.e., $n \geq 4$. Hence, $0 \leq \mathrm{codim}_{\mathrm{Gr}(2,5)}(X) \leq 2$.

(5°) **Arzhantsev** (2011): The homogeneous manifold $\mathrm{Gr}(2,5)$ is additive, and its linear sections as well (**Hwang–Fu** (2018)).

□

Uniqueness of additive structures:

- (**Fu-Hwang**, 2014): If X smooth additive Fano variety with $\rho(X)=1$ such that $X \ncong \mathbf{P}^n$, then the additive structure on X is unique.
- (**Dzhunusov**, 2022): Uniqueness criterion for additive toric varieties.

Several remaining issues in the case $\rho(X) = 1$

(A) What is the boundary divisor $\Delta = X \setminus \mathbf{A}^n$? (cf. Hirzebruch's problem)

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(B) What about **singular** varieties? (cf. Equivariant MMP)

(C) What if the ground field $k \neq \overline{k}$? (cf. $k = \mathbf{C}(Y)$ function field)

Dubouloz–Kishimoto–M., 2024

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Let k be a field of characteristic zero, and let Y be a k-form of X_L. Then, along the proof, is possible to take into account the action of $\mathrm{Gal}(\overline{k}/k)$ in order to analyze the existence of $\mathbf{G}_{a,k}^n$-structures on Y.

§4. Some ingredients for quintic del Pezzo varieties

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Additive refinement of Blanchard's Lemma (DKM, 2024)

Let $f : X \to Y$ a proper morphism with connected fibers between algebraic varieties, with $n = \dim(X)$ and $m = \dim(Y)$. Then,

- Any $\mathbf{G}_a^n$-structure on X induces a unique $\mathbf{G}_a^m$-structure on Y.

§5. Sketch of Proof

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The blow-up of of the indeterminacy locus $V_p \cong \mathbf{P}^3$ induces the following:

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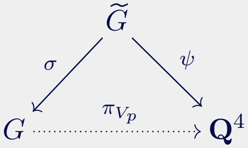

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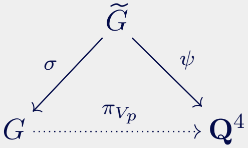

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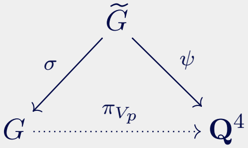

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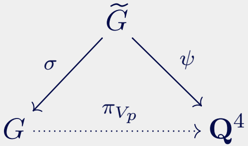

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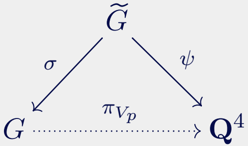

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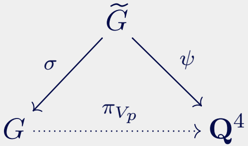

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Remark (Lie theory)

Using the fact that $\mathrm{Gr}(2,5) \cong \mathrm{GL}_5 / P$ is a rational homogeneous space, we can exhibit the desired $\mathbf{G}_a^6$-structure explicitly in Plücker coordinates.

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Moreover, among the infinite $\mathbf{G}_a^6$-structures on $\mathbf{P}^6$, the induced action on $\mathbf{P}^6$ is the naive (i.e. toric) one.

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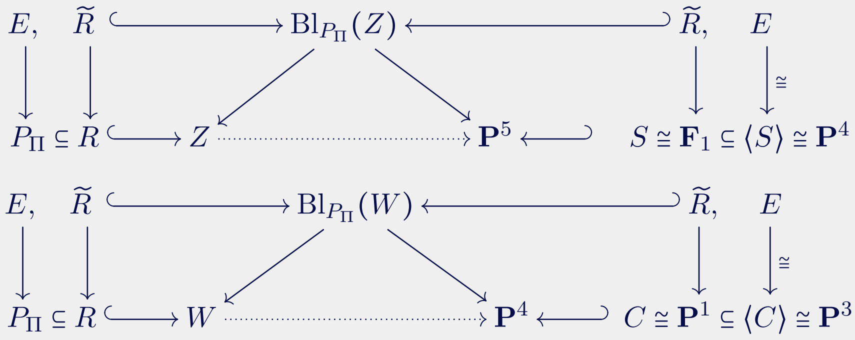

are equivariant, and induce the naive $\mathbf{G}_a^n$-structure on $\mathbf{P}^n$. □

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2. The explicit projective models obtained by Cheltsov–Prokhorov (2021) allow us to conclude. □

Thanks for your attention!